10-#1) Taylor Practice

1) (a) $f(x) = x\left(\frac{x}{3} - \frac{(x/3)^2}{2} + \frac{(x/3)^3}{3} - \frac{(x/3)^4}{4} + \ldots \frac{(-1)^{n+1}(x/3)^{n+1}}{n}\right)$

$= \frac{x^2}{3} - \frac{x^3}{18} \; \frac{x^4}{81} - \frac{x^5}{4 \cdot 3^4} + \ldots \frac{x^n}{n \cdot 3^n}(-1)^{n+1}$

b) $\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{3^n \cdot n}{x^n}\right|$

$\lim_{n\to\infty}\left|\frac{n\,x}{(n+1)3}\right| = \left|\frac{x}{3}\right|$

$\left|\frac{x}{3}\right| < 1$

$-1 < x/3 < 1$

$-3 < x \leq 3$

$x = +3 \quad \sum_{n=0}^{\infty} \frac{3}{n}(-1)^{n+1}$ conv.

$x = -3 \quad \sum_{n=0}^{\infty} 3/n$

$x = -3 \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-3)^{n+1}}{n \cdot 3^n}$

diverges

c) $|P_4(2) - f(2)| \leq \left|\frac{-2^5}{4 \cdot 3^4}\right|$

2) a) $\frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \ldots$

$4 + 5x - \frac{x^2}{2}$ $\qquad f(.2) \approx 4 + 5(.2) - \frac{.2^2}{2}$

$= 4.98$

b) $g(x) = f(x^3)$

$g(x) = 4 + 5x^3 - \frac{x^6}{2}$

$g'(x) = 15x^2 - \frac{6x^5}{2}$

c) $8 + \frac{3(x-1)}{1} - \frac{2(x-1)^2}{2} + \frac{3}{2}\frac{(x-1)^3}{3!}$ $\qquad \frac{3}{2 \cdot 3 \cdot 2}$

$8 + 3(x-1) - (x-1)^2 + \frac{1}{4}(x-1)^3$

d) error $\leq \left|\frac{300 \cdot (1.1 - 1)^4}{4!}\right|$

$\leq \left|\frac{300(1/10)^4}{4 \cdot 3 \cdot 2 \cdot 1}\right| = \frac{300}{4 \cdot 3 \cdot 2 \cdot 10^4} = \frac{100}{4 \cdot 2 \cdot 10^2 \cdot 10^2} = \frac{1}{800}$

error $\leq \frac{1}{800}$

3. $f(x) = \dfrac{1}{x^2+9}$

a) $\displaystyle\int_3^{\infty} \frac{1}{x^2+9}\,dx$

$\displaystyle\lim_{b\to\infty} \int_3^b \frac{1}{9(x^2/9+1)}\,dx$

$\displaystyle\lim_{b\to\infty} \frac{1}{9}\int_3^b \frac{1}{(x/3)^2+1}\,dx$ $\quad u = x/3$, $du = 1/3\,dx$, $3\,du = dx$

$\displaystyle\lim_{b\to\infty} \frac{3}{9}\int_1^b \frac{1}{u^2+1}\,du$

$\displaystyle\lim_{b\to\infty} \frac{1}{3}\left(\tan^{-1}(u)\right)\Big|_1^b$

$\displaystyle\lim_{b\to\infty} \frac{1}{3}\left[\tan^{-1}(b) - \tan^{-1}(1)\right]$

$\dfrac{1}{3}\left[\dfrac{\pi}{2} - \dfrac{\pi}{4}\right] = \dfrac{\pi}{6} - \dfrac{\pi}{12}$

$= \dfrac{2\pi}{12} - \dfrac{\pi}{12} = \boxed{\dfrac{\pi}{12}}$

b) $\displaystyle\sum_{n=\infty}^{\infty} \frac{1}{x^2+9}$ converges by the Integral since

$\left(\dfrac{1}{x^2+9}\right.$ is positive, decreasing and continuous$)$

c) $\displaystyle\lim_{n\to 1} \left| \frac{(n+1)^2+9}{e^{n+1}} \cdot \frac{e^n}{n^2+9} \right|$

$\displaystyle\lim_{n\to 1} \left| \frac{(n^2+2n+10)\cdot \cancel{e^n}}{\cancel{e^n}\cdot e \cdot (n^2+9)} \right| = \frac{1}{e} < 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+9)}{e^n}$ converges absolutely.

4. $\dfrac{\pi}{e} - \dfrac{\pi}{e^2} + \dfrac{\pi}{e^3} - \dfrac{\pi}{e^4} + \dots$ geometric $a = \pi/e$, $r = (-1/e)$

$= \dfrac{\frac{\pi}{e}}{1-(-1/e)} = \dfrac{\frac{\pi}{e}}{\frac{e+1}{e}} = \dfrac{\pi}{e}\cdot\dfrac{e}{e+1} = \dfrac{\pi}{e+1}$ $\quad \boxed{B}$

5. $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$e^{-x^2} = 1 + (-x^2) + \dfrac{(-x^2)^2}{2} + \dfrac{(-x^2)^3}{3!} + \dfrac{(-x^2)^4}{4!}$

$= 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!}$ $\quad \boxed{D}$

6. $f'(x) = \sin(x^2)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

$$\int \sin(x^2)\,dx = \frac{1}{3}x^3 - \frac{1}{7}\cdot\frac{x^7}{3!} + \frac{1}{11}\,\frac{x^{11}}{5!} - \dots \quad \boxed{D}$$

7. $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ $\boxed{C}$

~~(A)~~ Σ can't converge by comp. to $\Sigma\, 1/n$

$\frac{1}{\ln(n+1)} > \frac{1}{n}$ for all $n > 1$

~~(B)~~ I think series diverges so don't compare with $\Sigma\, 1/n^2$

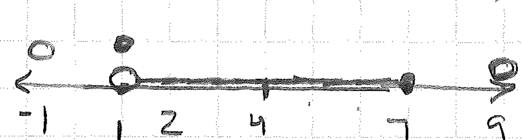
~~(D)~~ Σ cant diverge by comp to $\Sigma\, 1/n^2$

8. $\lim_{n\to\infty} a_n = \infty$ implies $\lim_{n\to\infty} \frac{1}{a_n} = 0$

$a_{n+1} \geq a_n \geq 0$ implies $\frac{1}{a_{n+1}} \leq \frac{1}{a_n}$ so $\sum_{n=1}^{\infty} \frac{1}{a_n}(-1)^n$ converges by AST

$\boxed{D}$

9. center = 4
radius is at least 3



$\boxed{B}$

end points unknown, convergence at $x=1$ unknown

10. $\frac{2}{n} = \frac{3}{100}$

$3n = 200$

$n = 66.67$

$$|S - P_{66}| < \left|(-1)^{67+1}\frac{2}{67}\right|$$

$$< 2/67 = 0.02985$$

If 66 terms are used the error will be less than the 67 term.

$\boxed{k = 66}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = S$

$$\frac{1}{\sqrt{n}} = \frac{3}{100}$$

$$3\sqrt{n} = 100$$

$$\sqrt{n} = 100/3$$

$$n = 1111.11\ldots$$

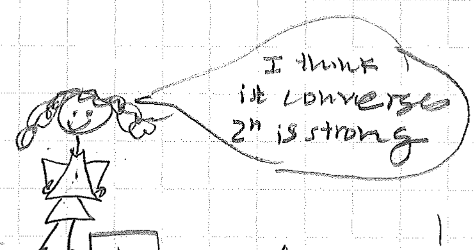
If 1111 terms are used the error will be less than the |1112 term.|

$$\left|\frac{(-1)^{1113}}{\sqrt{1112}}\right| = 0.029988\ldots \quad \boxed{C}$$

12. $$\frac{e^x + e^{-x}}{2} = \frac{\left(1 + x + x^2/2 + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + x^2/2 - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)}{2}$$

$\boxed{A}$ $$\frac{2 + \frac{2x^2}{2} + \frac{2x^4}{4!}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n}$



~~(A)~~ ~~(B)~~ ~~(C)~~ $\boxed{D}$

Lim must be 1

$$\lim_{n\to\infty} \frac{\frac{1}{2^n - n}}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - n}$$

$$= \lim_{n\to\infty} \frac{2^n \cdot \ln(2)}{2^n \cdot \ln(2) - 1} = 1$$